# Algebraic Behavior of Idempotent Elements in Near Ring

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**Abstract**

The study of Boolean near-rings and idempotent elements forms an important part of modern algebra, particularly in exploring the structural behavior of near-rings and their algebraic properties. In this work, we investigate the nature of idempotent elements in near-rings with special emphasis on near-fields and Boolean near-rings. We show that:

1.Near fields have exactly two idempotent elements. We claim that they are the trivial idempotent elements. Further, we give concrete example to support this claim that if R is not a near field, the idempotent elements are not exactly two.

2.Idempotent element is a near integral domain also trivial.

3.Every Boolean near ring is commutative near ring.

Overall, our findings provide a deeper understanding of the interplay between idempotent elements, near-fields, near integral domains, and Boolean near-rings. The results not only contribute to the theoretical advancement of near-ring theory but also establish useful connections with commutativity, algebraic simplicity, and structural classification. This study opens pathways for further exploration of near-rings in relation to other algebraic systems and their applications in both pure and applied mathematics.

Moreover, future research can extend these results to hybrid algebraic systems, examine categorical frameworks that unify near-rings with related algebraic structures, and explore their interdisciplinary use in computer science, data encryption, and logical reasoning models, thereby strengthening both the theoretical and applied dimensions of near-ring theory.

**Keywords**

Commutative Near Ring, Idempotent Elements, Boolean Near Ring, Near Field

## 1. History of Near Ring

In late 1968, approx. fifty-five years ago, the first conference on near rings and near fields took place in the Mathematics Forschung Institute Oberwolfach- Germany. In this conference, mainly present historical survey on “The beginning and development of near-ring theory”.

The theory of near-rings emerged as a response to the need for algebraic structures that model real-world systems more flexibly than traditional rings. The concept has its roots in the evolution of abstract algebra during the late 19th and early 20th centuries. Mathematicians sought structures that would extend or generalize classical ring theory, particularly in contexts where certain axioms (like the commutativity of addition or the left distributive law) were too restrictive or simply not satisfied. The formal definition of a ring, as an algebraic structure equipped with two binary operations (addition and multiplication), was influenced heavily by the work of Dedekind, Noether, Hilbert, and others in the late 1800s and early 1900s. Rings provided a unified framework for solving problems in number theory, linear algebra, and algebraic geometry. As algebra developed, it became increasingly clear that many natural structures encountered in mathematics and related disciplines exhibited almost ring-like behavior but did not fully satisfy all ring axioms. For instance, function spaces, endomorphism sets, and certain transformation sets displayed non-commutative addition or lacked one of the distributive laws.

Near-rings are one of the generalized structures of rings. The term 'near-ring' was first introduced in a more structured way in the mid-20th century, particularly in the 1940s and 1950s, when mathematicians like Günther Pilz, B.H. Neumann, H. Neumann, and others began to formalize systems that partially satisfied ring axioms. Their work focused primarily on relaxing the left distributive law and removing the requirement of commutative addition, thereby forming a new class of algebraic structures that could accommodate a wider variety of mathematical objects. The research and study on near rings is very systematic and continuous. The foundational groundwork for near-rings was laid by the study of rings, groups, and semigroups.

B.H. Neumann and H. Neumann were among the first to explore algebraic systems that arose naturally in group theory and transformation semigroups, which lacked symmetry in distribution or addition. Their work highlighted that these algebraic structures, though not rings, still exhibited enough internal consistency to be studied as a distinct class.

Their introduction of the gamma near-ring was a pivotal moment in the evolution of the theory. In a gamma near-ring, the binary operations were defined in a context involving an external set of operations (often denoted as Γ), which acted on the elements in a non-traditional manner. This provided an elegant generalization of near-rings and created a rich field of study that combined group theory, semigroup theory, and transformation theory.

Günther Pilz, Austrian mathematician made significant contributions to the development and classification of near-rings. His work during the 1960s and 1970s was instrumental in formalizing the theory and exploring its applications in endomorphism rings, modules, and automata theory. Pilz also authored one of the foundational books in the area, Near-Rings: The Theory and Its Applications, which provided a systematic treatment of near-ring theory, encompassing its axioms, examples, homomorphisms, ideals, and applications. His work not only unified the fragmented research of earlier decades but also inspired a new generation of mathematicians to study near-rings as a legitimate and fruitful algebraic field.

Motivation Behind the Concept for developing near-ring theory stemmed from the realization that many algebraic systems particularly those arising in applied contexts-did not conform to the strict requirements of rings. Specific motivations included: modeling non-linear operations, transformation functions, non-commutative settings, and incomplete distributivity.

In the period of 1960s–1980s, , near-ring theory began to flourish. Several developments occurred during this period: generalizations, module theory, radical theory, categorical approaches, and combinatorics and geometry. In 1990s till Present, near rings are being used since the development of Calculus, but the basics and key ideas to formalized in 1905 by Dickson's who defined the near field to give examples of Non- desarguesian planes. In 1930, Wieland studied near rings, which were not near fields. Further text material about subject can be found from two famous books on the near-rings [1,2]. The scope of near-ring theory has continued to expand in both theoretical and applied directions. Fuzzy near-rings, near-ring-based cryptography, topological near-rings, computational tools, and interdisciplinary applications are now common in the field.

In last, the abstraction of rings and groups to its present-day applications in computational science and discrete mathematics, near-ring theory has evolved significantly over the past century. Key figures like the Neumanns and Günther Pilz played a foundational role in defining and developing near-ring theory, while modern mathematicians have expanded its boundaries into new domains. The historical development of near-rings showcases not just a logical progression of algebraic thought but also an enduring effort to make mathematics applicable to a broader range of problems and systems.

In algebra, the concept of idempotent elements has been extensively studied due to their central role in understanding the internal structure and functional decomposition of algebraic systems. An element e of a near ring N is called idempotent if $e^2 = e$ . This definition resembles that found in rings and semigroups, but in nearrings, the properties and implications of idempotent elements are notably more intricate due to the relaxed distributive laws. Specifically, near rings typically satisfy only right distributivity, i.e., $(a+b)c = ac+bc$ , and not necessarily left distributivity. This distinction makes the identification and analysis of idempotent elements in nearrings both challenging and intellectually stimulating.

Idempotent elements serve as a powerful tool in the structural analysis of algebraic systems. In near rings, they often define specific substructures or subnear rings that exhibit simplified or well-understood behaviors. For instance, if a near ring contains a large number of idempotent elements, these may partition the near ring into disjoint components or help characterize ideals and congruence relations. Moreover, the set of idempotent elements can sometimes be used to construct projection maps or define endomorphisms with desirable properties, such as being homomorphic images or retrievable by idempotent actions. This property is particularly useful in the study of transformation near rings, where functions operate on sets and the composition structure benefits from the presence of idempotent transformations.

A key focus in near ring theory is to classify and characterize all idempotent elements within given classes of near rings. Various methods have been developed to identify idempotent, including solving algebraic equations of the form $x^2 = x$ under near ring operations, studying fixed points of self-maps, or using matrix representations where applicable. The presence of idempotent elements also has implications for the decomposition of modules over near rings. In analogy to ring theory, where idempotent lead to direct sum decompositions of modules, a similar-though generally more restricted-phenomenon may be observed in near ring-modules. In fact, certain near rings admit a structure where the presence of orthogonal idempotent enables a type of partial decomposition that mimics behavior found in semi-simple structures.

Idempotent elements are also instrumental in understanding the lattice of ideals and sub near rings. For example, in zero-symmetric near rings (where 0. $a = 0$ for all $a \in \mathrm{N}$ ), idempotent often generate minimal right ideals, thereby providing insight into the layering of the ideal structure. In practical applications, especially in automata theory and formal language processing, idempotent transformations help define stable states or repeated configurations. These features make idempotent elements a key topic not only in theoretical algebra but also in applied mathematical contexts.

Further research in this area continues to explore the role of idempotent in non-traditional or generalized near rings, including fuzzy near rings, interval near rings, and near rings over groups with additional structures. The broad applicability and the structural insights offered by idempotent elements confirm their importance in advancing both the foundational theory and the practical uses of near ring algebra.

Secondly we discuss commutative near ring. Near rings generalize rings by relaxing one or more ring axioms, and among these generalizations, the concept of a commutative near ring presents an interesting hybrid structure. A near ring N is said to be commutative if the multiplication operation satisfies $a \cdot b = b \cdot a$ for all $a, b \in \mathrm{N}$ . Unlike in ring theory, however, a commutative near ring does not necessarily possess two-sided distributivity; typically, only the right distributive law holds. This subtle distinction allows for a broader class of algebraic systems while still retaining enough structure to enable deep mathematical analysis.

The study of commutative near rings is motivated by the desire to understand how the relaxation of distributivity affects the classical results from commutative algebra. In a commutative ring, many theorems rely on the interplay between commutativity and full distributivity, but in a commutative near ring, these theorems may not hold in full generality or may require additional conditions. For example, the existence and uniqueness of factorization into irreducible elements-a cornerstone of ring theory-is generally more complicated in near rings. Likewise, the structure of ideals, maximal and prime elements, and modules over a commutative near ring may differ significantly from their ring-theoretic counterparts.

One of the key aspects of commutative near rings is their potential to model algebraic behavior in systems where symmetry in multiplication is desired, but full distributivity is not guaranteed. This is particularly relevant in certain areas of computer science, coding theory, and cryptography, where operations may be inherently asymmetric in terms of addition and multiplication. By studying commutative near rings, researchers can explore systems that strike a balance between complexity and tractability. These structures also lend themselves to analysis via polynomial functions, transformation semigroups, and matrix operations under constrained conditions.

Another important area of interest is the homomorphic images and substructures of commutative near rings. Given a commutative near ring, one can study its endomorphism near ring, which often retains commutativity under composition or application. Moreover, questions regarding the existence of multiplicative identities, zero divisors, and unit elements in commutative near rings add layers of richness to their classification. Some commutative near rings have been constructed explicitly from modules over commutative rings, while others emerge as functional near rings acting on vector spaces, semigroups, or even sets with additional algebraic constraints.

The classification and construction of commutative near rings remain an active area of research. For example, one can define commutative near rings by restricting the operations in well-known ring structures or by constructing quotient structures over more complex algebraic systems. In this sense, commutative near rings serve as both a theoretical generalization of rings and a practical modeling tool for systems with limited distributive properties but symmetric multiplicative interactions. Their intermediate position between general near rings and full rings provides valuable insight into algebraic hierarchies and their implications for both pure and applied mathematics.

Boolean near rings form a specialized and highly structured class of near rings in which every element is idempotent, that is, for each element $a \in \mathrm{N}$ , we have $a^2 = a$ , and the additive structure of the near ring is typically that of a Boolean group i.e., an abelian group where every element is its own inverse. This means $a + a = 0$ for all $a \in \mathrm{N}$ , which implies that every element has order 2 under addition. Such near rings are deeply connected to logic, switching theory, and the algebraic foundations of digital circuits.

The motivation for studying Boolean near rings stems from their close alignment with the structure of Boolean algebras and logical systems. In fact, Boolean near rings can be seen as a non-classical algebraic framework that captures binary-state behaviors, such as on-off switching or true-false logic. These structures are particularly useful in modeling finite-state machines, decision trees, and control systems. Because every element is idempotent under multiplication, Boolean near rings exhibit a high degree of predictability and stability, which makes them especially suitable for modeling repetitive or stabilized systems.

One of the most intriguing aspects of Boolean near rings is their simplicity combined with expressive power. Despite having only two possible values for each element under addition and a constrained multiplication operation, these near rings can be used to represent complex logical expressions and operations. For instance, in digital logic design, Boolean near rings can be used to describe networks of logic gates, where each gate corresponds to a specific idempotent operation. This algebraic representation helps in simplifying and optimizing logical expressions, particularly in the design and analysis of integrated circuits.

The structural analysis of Boolean near rings also offers insights into decomposition theories and automorphism groups. Since every element is idempotent, Boolean near rings tend to have flat or degenerate ideal structures, which nonetheless can be useful in classification problems. The simplicity of their operation allows for complete enumeration of elements and exhaustive study of substructures. Furthermore, Boolean near rings can be realized as function near rings on finite sets, where the multiplication is defined via function composition and the addition corresponds to pointwise XOR operations. This realization links Boolean near rings directly to computer science, especially in areas related to parallel processing, fault-tolerant systems, and artificial intelligence models that rely on binary-state logic.

Mathematically, Boolean near rings are often studied in relation to Boolean rings and lattices. While Boolean rings also have the property that every element is idempotent under multiplication, the difference lies in the extent of distributivity and the algebraic context. Boolean near rings allow researchers to explore what happens when the distributive law is partially or selectively applied, opening up new theoretical questions and applications. This makes Boolean near rings a flexible yet powerful algebraic structure that extends the reach of classical Boolean logic into broader mathematical and computational domains.

In conclusion, Boolean near rings are not just a mathematical curiosity but a robust framework for modeling logical systems, analyzing algebraic structures with constrained operations, and bridging the gap between abstract algebra and digital applications. Their utility in representing and simplifying binary operations makes them an essential tool in both theoretical research and practical implementations in modern technology.

## 2. Introductory Material

We call an empty set R a right near ring if (R, +) is group, not necessarily abelian, (R,.) is a semi group and multiplication is right distributive over addition. The following concepts are as in defined in the ring theory: (right) identities, (right) cancelable elements and (right) zero divisors. If (R, .) is commutative then we call R is a commutative near ring. If all non-zero elements R are right cancelable then R fulfills the right cancelation law.

A near ring without zero divisors is called near integral domain. If (R*(=R\{0}),.) is also a group then R is called a near field.

If R is a Boolean near ring, then $x^2 = x$ .

All introductory material on near rings can be found in the writing [1,2].

## 3. Scope and Applications of Near-Ring

Over the decades, the theory of near-rings has evolved from a purely abstract endeavor into a domain with numerous applications across mathematics and theoretical computer science. The following sections outline key areas where near-rings play a pivotal role:

### 3.1 Algebraic Structures and Generalizations

Near-rings serve as generalized algebraic structures bridging gaps between rings, semigroups, and groups. They are especially useful in the classification and study of non-associative or partially distributive systems. In particular, gamma near-rings**,** generalized near-rings**,** and near-fields (which relax field axioms) form an active area of study in algebra.

### 3.2 Combinatorics and Group Actions

Near-rings have applications in combinatorial design theory, especially in the construction of block designs and difference sets. Near-rings act on groups, allowing the formulation of group actions that are not necessarily linear, thus broadening the analytical framework of permutation groups and symmetry.

### 3.3 Automata and Formal Language Theory

In automata theory, near-rings model the behavior of state transition functions. The non-commutative addition and relaxed distributivity naturally fit the behavior of deterministic and non-deterministic automata, where inputs result in transitions governed by state and context rather than linear transformations.

### 3.4 Coding and Cryptography

Near-rings are used in the construction of non-linear codes and error-correcting systems. Their non-commutative nature makes them suitable for modeling non-linear encryption schemes where ring structures are too restrictive. Near rings can also produce rich algebraic invariants that support authentication and message integrity. Near ring also give the new horizon to modern algebra or abstract algebra.

### 3.5 Topological and Geometrical Structures

Topological near rings extend near-ring concepts into topology. They are used in the study of topological groups, module structures, and topological vector spaces. In geometry, near-rings model geometrical transformations and support research into finite geometries and affine planes.

### 3.6 Computer Science and Logic

The logic of computer programs, particularly involving non-deterministic computation, finds near-ring algebra useful. Programs modeled as transformations on state spaces under function composition naturally form near-ring structures. Also, semantics of programming languages especially those involving side effect are represented using near ring-like structures.

**4. Research Frontiers and Current Trends**

The field of near-ring theory is expanding with research addressing both theoretical and applied problems. The current trends include:

Fuzzy **Near-Rings**: Combining fuzzy set theory with near-ring algebra to manage uncertainty in algebraic computations and AI systems.

Intuitionistic and **Rough Near-Rings**: Integrating logic-based uncertainty frameworks into algebra, useful in decision sciences.

Categorical **Approaches**: Applying category theory to near-rings to better understand homomorphisms, modules, and factorial properties.

Near**-Ring Modules**: Investigating **near-ring modules**, which generalize ring modules, enabling algebraic modeling of more complex operations.

Homological **Algebra**: Some researchers are exploring homological aspects of near-rings, such as extensions, derivations, and cohomology theories.

Near**-Ring Radicals**: The study of radical theory in near-rings, such as Jacobson radicals and prime radical properties, mirrors developments in ring theory.

**5. Educational Value**

In educational contexts, near-rings serve as excellent tools for:

•Developing abstract reasoning,

•Exploring the boundaries of algebraic systems,

•Constructing counterexamples in ring theory,

•Bridging group and semigroup theory.

**6. Comparative View: Near-Rings Vs Rings**

| Ring [3,4] | Near-Ring[1,2] | Property[5,6] |
|---|---|---|
| Abelian | Not necessarily abelian | Additive Group |
| Both left and right | Usually only right distributive | Distributive Laws |
| Associative with identity | Associative, identity optional | Multiplicative Structure |
| Broad, especially linear | Broader in non-linear systems | Applications |
| Required | Not Required | Commutativity (Addition) |

**7. Near Rings Results**

In this section we present three facts related to idempotent elements in near field, near integral domain and Boolean near ring.

**7.1 Theorem**

Near field have exactly two idempotent elements. Proof:

As we know that

$0^2 = 0$, we only need to prove the result for non-zero elements. Let $a \in \mathrm{R}$, such that $a \neq 0$. If a is an idempotent element then,

$$\begin{aligned} a &= a^2 \\ \Rightarrow a + (-a) &= a^2 + (-a) \\ \Rightarrow 0 &= a^2 - a \\ \Rightarrow 0 &= a(a-1) \end{aligned}$$

Either $a = 0$ or $a - 1 = 0$

Since $a$ is non zero therefore $a = 1$.

Next we give example to show that they are the only elements in $\mathrm{R}$ as elements other than 0 and 1 are not idempotent element in near field [7,8].

**Example-1:**

( $Z3$ , +, .) Under addition modulo 3 and multiplication modulo 3 is near field [9].

| ● | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |

0 and 1 are the only idempotent elements as $0^2=0$, $1^2=1$ and $2^2\neq 2$.

Question arises how many idempotent elements are in R if R is not near field.

**Example-2:**

( $Z4$ , +, .) is not near field [10].

| ● | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 1 | 1 |
| 3 | 0 | 2 | 0 | 2 |

We note that $0^2=0$ and $1^2=1$ and $2^2=2$ which shows that 0, 1, 2 are idempotent elements. Hence if R is not a near field then idempotent elements are not just the trivial elements.

**7.2 Theorem**

The idempotent elements are a near integral domain trivial. Proof:

Suppose R is a near integral domain. $x \in \mathrm{R}$ and $x$ is an idempotent element. Then $x^2 = x$

So, $x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

Since R is an integral domain, $x = 0$ or $x = 1$ .

Hence it is proved an integral domain have trivial idempotent elements.

**7.3 Theorem**

Every Boolean near ring is commutative near ring.

Proof:

Suppose R is Boolean near ring then first we have to prove

i) $x + x = 0, \quad \forall x \in \mathrm{R}$

ii)$x$=$y$ $x + y = 0$ then $x = y$

For first part Let

$x \in \mathrm{R}$

$x + x \in \mathrm{R}$ , since ( R , +) is group.

$\Rightarrow (x+x)^2 = x + x$ , since R is Boolean near ring

$\Rightarrow (x+x)(x+x) = x + x$

$\Rightarrow x^2 + x^2 + x^2 + x^2 = x + x$

$\Rightarrow x+x+x+x = x+x$ , by left cancellation law

$\Rightarrow x+x = 0, \quad \forall x \in \mathrm{R}$

For Second Let $x+y=0$

then $x+y = x+x$ by part (i)

$y = x$ by left cancellation law

Now, we proof R is commutative near ring. Let R is Boolean near ring.

$x, y \in \mathrm{R}$ then $x+y \in \mathrm{R}$ , Since ( R , +) is group

$\Rightarrow (x+y) = (x+y)^2$

$\Rightarrow (x+y) = (x+y)(x+y)$

$\Rightarrow (x+y) = x^2 + xy + yx + y^2$

$\Rightarrow (x+y) = x + xy + yx + y$ as $x^2 = x$ or $y^2 = y$

$\Rightarrow xy + yx = 0$ by cancelation laws

$\Rightarrow xy = yx$ by above $x = y$

Hence, Every Boolean near ring is commutative near ring.

The bibliography of near rings is available on the website [11].

## 8. Conclusion

The theory of near-rings originated as a natural generalization of ring theory, relaxing certain axioms to accommodate algebraic systems arising in real-world phenomena, particularly in transformations, automata, and non-linear structures. Since its formal inception in the mid-20th century, the field has grown substantially, both in theoretical richness and in practical applications. Near-rings bridge the gap between abstract algebra and applied domains such as coding theory, computer science, logic, and combinatorics. Their flexible structure allows researchers to construct and study systems that resist classical algebraic formalism. As mathematical research moves toward interdisciplinary and application-driven exploration, the importance and scope of near-rings are only expected to grow.

In last, whether in modeling non-linear systems, designing cryptographic protocols, or understanding the internal logic of computational processes, near-rings offer a versatile and profound algebraic framework. Continued research in this domain promises not only to enrich algebra but also to impact various branches of science and technology.

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