Fokker-Planck Equation in a Astrophysical Plasma Dynamics Model and Its Property Analysis

Junyu Luo*, Jing He

College of Physics and Electronic Science, Hubei Normal University, Huangshi, 435002, China

*Corresponding author's email: 18173957783@163.com

Abstract

The propagation of plasma beams, especially those from stellar explosions, plays a crucial role in shaping the dynamics of interstellar media, cosmic ray acceleration, and galaxy evolution. This study presents a comprehensive mathematical model describing the evolution of these plasma beams in galactic environments, incorporating deterministic forces from gravity and electromagnetism alongside random interactions with turbulent interstellar media. We derive a generalized Fokker-Planck equation that describes the distribution of particles in phase space, extending the classical Liouville equation to account for diffusion processes in momentum space. The equation includes the effects of weak-field gravitational forces, electric fields, magnetic fields, as well as random perturbations arising from plasma turbulence and magnetic fluctuations. We rigorously analyze the mathematical properties of the equation, proving the existence, uniqueness, and stability of weak solutions. Additionally, we derive key conservation laws describing the particle number, momentum, and energy, and investigate the conditions under which these quantities are conserved or dissipated. In equilibrium states, the particle distribution is shown to converge to a form analogous to the Maxwell-Boltzmann distribution, emphasizing the connection between plasma dynamics and classical statistical mechanics. Our study provides a unified framework for understanding plasma transport in astrophysical environments, offering profound insights into phenomena such as cosmic ray propagation and the evolution of supernova remnants. This research lays an important foundation for future studies on the interactions between particles, electromagnetic fields, and turbulent media in astrophysical settings.

Keywords

Fokker-Planck Equation, Plasma Dynamics, Astrophysics, Partial Differential Equation, Conservation Laws, Equilibrium Distribution

1. Introduction

The propagation of plasma beams ejected from stellar explosions through galactic environments represents a fundamental process in astrophysics, influencing the dynamics of interstellar medium, cosmic ray acceleration, and galaxy evolution. In such scenarios, the plasma beam, composed of relativistic charged particles, interacts with the weak-field general relativistic effects near the stellar source, electromagnetic fields pervasive in the galaxy, and a myriad of stochastic perturbations arising from turbulent interstellar media, magnetic fluctuations, and gravitational microlensing. To model this complex propagation, we adopt a statistical approach by deriving a partial differential equation (PDE) that incorporates deterministic forces from gravity and electromagnetism alongside diffusive terms representing random disturbances. The foundational framework is the Fokker-Planck equation, which extends the collisionless Liouville equation to account for stochastic interactions in phase space. This equation has been extensively utilized in plasma physics and astrophysics to describe particle transport under combined deterministic and random influences. For instance, early works on cosmic ray diffusion in galactic magnetic fields laid the groundwork for such models [1,2]. Subsequent studies incorporated relativistic effects in plasma dynamics [3,4], while general relativistic corrections in weak-field approximations have been explored in contexts like neutron star atmospheres [5,6]. In galactic scales, the plasma beam's spread is affected by stochastic scattering from interstellar turbulence [7,8], leading to diffusive behavior in momentum space. Electromagnetic fields, including large-scale galactic magnetic fields, induce Lorentz forces that alter particle trajectories [9,10]. Gravitational effects, though weak, become pertinent near the stellar explosion site [11,12]. The inclusion of source terms accounts for particle injection from the stellar burst [13,14]. Mathematical analyses of the Fokker-Planck equation in these settings have focused on existence and uniqueness of solutions [15,16], stability properties [17,18], and conservation laws [19,20]. Non-negativity of the distribution function ensures physical consistency [21,22]. Equilibrium distributions often relate to Maxwell-Boltzmann forms under thermal balance conditions [23,24], though relativistic and magnetic effects impose stringent constraints [25,26]. Recent advancements have applied these models to specific astrophysical phenomena, such as gamma-ray burst afterglows [27,28] and supernova remnant shocks [29,30]. By integrating these elements, our analysis provides a unified PDE description of the plasma beam's evolution, elucidating its mathematical properties and physical implications.

2. Model Formulation and Equation Derivation

2.1 Phase Space Distribution Function

In statistical physics, to describe the microscopic behavior of a macroscopic particle system, we introduce the phase space distribution function $f(\mathbf{x},\mathbf{p},t)$ [31,32]. This function is a probability density, representing the number density of particles near position $\mathbf{x}$ and momentum $\mathbf{p}$ at time t per unit phase space volume. For systems with numerous particles, tracking individual deterministic trajectories is infeasible. Thus, the distribution function offers a statistical description, enabling the study of macroscopic properties and evolutionary laws[7]. The phase space is typically six-dimensional, comprising three-dimensional position coordinates $\mathbf{x}=(x^1,x^2,x^3)$ and three-dimensional momentum coordinates $\mathbf{p}=(p^1,p^2,p^3)$.

2.2 From Liouville Equation to Fokker-Planck Equation

Particles in classical mechanics follow Hamilton's equations. For a collisionless particle system under deterministic external fields, the evolution of the phase space distribution function f is governed by the Liouville equation. This equation is essentially the continuity equation for particle number density in phase space, asserting that the phase space "fluid" is incompressible, with phase space volume elements remaining constant during motion (Liouville's theorem)[17].

The total derivative form of the Liouville equation is:

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{d\mathbf{x}}{dt}\cdot\frac{\partial f}{\partial \mathbf{x}}+\frac{d\mathbf{p}}{dt}\cdot\frac{\partial f}{\partial \mathbf{p}}=0 \quad (1)$$

where $\frac{d\mathbf{x}}{dt}=\mathbf{v}$ is the particle velocity, and $\frac{d\mathbf{p}}{dt}=\mathbf{F}$ is the force on the particle.

For a single relativistic particle, the velocity $\mathbf{v}$ relates to momentum $\mathbf{p}$ as $\mathbf{v}=\frac{\mathbf{p}}{\gamma m}$, where $\gamma=\sqrt{1+(|\mathbf{p}|/mc)^2}$ is the Lorentz factor, m is the rest mass, and c is the speed of light [4,25].

The force $\mathbf{F}$ can include various physical sources. In a general relativistic background, we consider the following main forces:

1.General relativistic gravity $\mathbf{F}_{GR}$: This arises from background spacetime curvature [5,33], reducing to Newtonian gravity in the weak-field limit.

2.Electric field force $q\mathbf{E}$: Force on charged particle q in electric field $\mathbf{E}$.

3.Lorentz magnetic field force $q(\mathbf{v}\times\mathbf{B})$: Force on charged particle q in magnetic field $\mathbf{B}$, with $\mathbf{v}$ the particle velocity[5].

Substituting these forces component-wise into the Liouville equation and expanding the summation notation yields the classical Liouville equation in detail:

$$\frac{\partial f}{\partial t}+\frac{p^i}{\gamma m}\frac{\partial f}{\partial x^i}+\left(F_{GR}^i+qE^i+q\epsilon^{ijk}\frac{p^j}{\gamma m}B^k\right)\frac{\partial f}{\partial p^i}=0 \quad (2)$$

However, real particle systems are not ideal collisionless systems. Microscopic interactions between particles or with background media (such as plasma or electromagnetic waves) introduce randomness. These random interactions cause stochastic changes in particle momentum (diffusion) and average energy loss (friction). The Fokker-Planck equation introduces a collision term on the right-hand side of the Liouville equation to statistically describe the impact of these random processes on the distribution function's evolution [15].

The classical collision term in the Fokker-Planck equation typically includes a drift (friction) term and a diffusion term. For Brownian motion, the momentum space collision term is often written as $\frac{\partial}{\partial p^i}\left(A^{ij}f\right)+\frac{\partial}{\partial p^i}\left(D^{ij}\frac{\partial f}{\partial p^j}\right)$ [3], where A^{ij} is the momentum space drift or friction coefficient, and D^{ij} is the momentum space diffusion coefficient.

Following the given form, we combine friction and diffusion effects into a generalized diffusion term $\frac{\partial}{\partial p^i}\left(K^{ij}\frac{\partial f}{\partial p^j}\right)$.

This implies that K^{ij} is a comprehensive diffusion coefficient matrix encompassing all momentum space random processes (including friction and diffusion). Although this form is uncommon in some contexts (where friction is typically $\frac{\partial}{\partial p^i}(\text{force}\times f)$), we adhere strictly to the given equation form.

Ultimately, we obtain the Fokker-Planck equation to analyze:

$$\frac{\partial f}{\partial t}+\frac{p^i}{\gamma m}\frac{\partial f}{\partial x^i}+\left(F_{GR}^i+qE^i+q\epsilon^{ijk}\frac{p^j}{\gamma m}B^k\right)\frac{\partial f}{\partial p^i}=\frac{\partial}{\partial p^i}\left(K^{ij}\frac{\partial f}{\partial p^j}\right)+S(\mathbf{x},\mathbf{p},t) \quad (3)$$

where K^{ij} is a generalized momentum space diffusion coefficient matrix, and $S(\mathbf{x},\mathbf{p},t)$ is the source or sink term for particles in phase space. For subsequent discussion, we introduce shorthand notation:

Particle velocity: $v^i=\frac{p^i}{\gamma m}$

Total deterministic force on the particle: $\mathcal{F}^i(\mathbf{x},\mathbf{p})=F_{GR}^i+qE^i+q\epsilon^{ijk}v^jB^k$ Using these symbols, the equation can be more concisely expressed as:

$$\frac{\partial f}{\partial t}+v^i\frac{\partial f}{\partial x^i}+\mathcal{F}^i\frac{\partial f}{\partial p^i}=\frac{\partial}{\partial p^i}\left(K^{ij}\frac{\partial f}{\partial p^j}\right)+S(\mathbf{x},\mathbf{p},t) \quad (4)$$

2.3 Physical Significance of Equation Terms

Let us analyze the physical significance of each term in the equation in detail:

1. $\frac{\partial f}{\partial t}$: This is the time evolution term of the distribution function. It describes the rate of change of particle number density at a point $(\mathbf{x},\mathbf{p})$ in phase space with time. If the system is in steady state or equilibrium, this term is zero.

2. $v^i\frac{\partial f}{\partial x^i}$: This is the spatial convection term (or advection term). It describes the change in the distribution function in phase space due to particles flowing from one region to another in space because of their macroscopic velocity $\mathbf{v}$. This term reflects the collective motion of particles in real space.

3. $\mathcal{F}^i\frac{\partial f}{\partial p^i}$: This is the momentum convection term (or drift term). It describes the change in the distribution function in phase space due to deterministic changes in momentum caused by the deterministic force $\mathcal{F}^i$. This represents the ”drift” or acceleration/deceleration of particles in momentum space.

- F_{GR}^i : Gravity induced by general relativistic effects (e.g., background spacetime curvature), reducible to Newtonian gravity in the weak-field approximation.

- qE^i : Force exerted by the electric field $\mathbf{E}$ on the charged particle q.

- $q\epsilon^{ijk}v^jB^k$: Force exerted by the Lorentz magnetic field $\mathbf{B}$ on the charged particle q. This force is always perpendicular to the particle velocity, so the magnetic field does no work but changes the momentum direction.

4. $\frac{\partial}{\partial p^i}\left(K^{ij}\frac{\partial f}{\partial p^j}\right)$: This is the momentum space diffusion term. It describes the disordered ”diffusion” or ”dispersion” of particle momentum in phase space due to microscopic random processes such as inter-particle collisions, thermal fluctuations in the background medium, and random electromagnetic fields. This term is typically the source of entropy increase in the system, smoothing the distribution function from local sharp structures to flat and uniform. The positive

definiteness of the coefficient matrix K^{ij} is the key physical condition ensuring it is a diffusion process (rather than anti-diffusion). It reflects dissipation and randomness within the system.

5. $S(\mathbf{x},\mathbf{p},t)$: This is the source/sink term. It represents the increase or decrease in particle number density in phase space due to non-interaction processes (such as particle generation, annihilation, injection, or removal in specific position or momentum ranges). For example, new particle production in nuclear reactors or particle absorption by walls in plasma can be described by source-sink terms.

3. Mathematical Properties and Classification of the Equation

The equation is a linear partial differential equation. Its highest-order derivative is the second-order derivative with respect to momentum $\mathbf{p}$, while the time derivative is first-order.

3.1 PDE Classification

Based on the nature of the highest-order derivative terms, the Fokker-Planck equation belongs to the class of parabolic partial differential equations. This classification shares mathematical structures similar to the heat conduction equation and the Schrödinger equation[34].

Specifically, the diffusion term $\frac{\partial}{\partial p^i}\left(K^{ij}\frac{\partial f}{\partial p^j}\right)$ formally resembles the Laplace operator (or a more general elliptic operator), and its positive definiteness imparts diffusion effects in momentum space. The presence of the time derivative $\frac{\partial f}{\partial t}$ makes it an evolution equation, describing the system's dynamic development over time. Parabolic equations are characterized by information propagating from initial states to future times, with infinite propagation speed (though actual physical effects are speed-limited).

3.2 Dissipative Nature

The presence of the diffusion term endows the equation with inherent dissipative properties. It tends to smooth the particle distribution in phase space, reducing sharp gradients and local aggregations, thereby increasing the system's entropy[22]. This process physically corresponds to the evolution of the particle system toward equilibrium or a more disordered state. In the absence of source-sink terms and deterministic forces, momentum diffusion homogenizes the particle momentum distribution.

3.3 Initial and Boundary Conditions

To uniquely determine the solution of the equation, in addition to the equation itself, we need to specify appropriate initial and boundary conditions[35].

-Initial condition: $f(\mathbf{x},\mathbf{p},0)=f_0(\mathbf{x},\mathbf{p})$. This specifies the particle distribution in phase space at the initial time $t=0$.

-Boundary conditions: The choice of boundary conditions depends on the specific physical problem and the definition of the phase space domain Ω.

-Homogeneous Dirichlet boundary condition $\left(f\,|_{\partial\Omega}=0\right)$: This means that when particles reach the boundary of the phase space domain Ω , they are immediately removed or absorbed from the system. This is common in systems describing particle escape or collection.

-Homogeneous Neumann boundary condition $\left(\mathbf{n}\cdot K\nabla_{\mathbf{p}}f=0\right)$: This indicates no particle flux through the boundary. It means particles cannot cross the boundary, which is reflective.

-Periodic boundary conditions: If the system is periodic in space or momentum, periodic boundary conditions can be imposed.

-Decay at infinity: For problems in unbounded domains $\mathbb{R}^6$, it is usually required that f tends to zero at infinity, and its derivatives decay sufficiently fast.

4. Existence, Uniqueness, and Stability of Solutions

We employ the Galerkin method, commonly used in functional analysis and partial differential equation theory [34], to prove the existence of weak solutions, and use energy methods to prove uniqueness and stability.

4.1 Problem Setup and Weak Formulation

We seek weak solutions to the equation in a bounded, smooth open phase space domain $\Omega \subset \mathbb{R}^6$ over the time interval $[0,T]$.

Basic assumptions:

1.Smoothness and boundedness of coefficients: The convection coefficients $e^i = \frac{p^i}{\gamma_m^i}$ and $\mathcal{F}^i(\mathbf{x},\mathbf{p})$ are bounded and sufficiently smooth on Ω. The momentum diffusion coefficient matrix $K^{ij}(\mathbf{x},\mathbf{p})$ is C^1 continuous and bounded on Ω.

2.Positive definiteness of diffusion coefficients: The matrix K^{ij} is symmetric, i.e., $K^{ij} = K^{ji}$. Moreover, there exists a positive constant $\lambda > 0$ such that for any real vector $\boldsymbol{\xi} \in \mathbb{R}^3$, $K^{ij}\xi^i\xi^j \geq \lambda |\boldsymbol{\xi}|^2$. This ensures the diffusion process is physically acceptable (dissipative)[22].

3.Source term: $S(\mathbf{u},t) \in L^2(\Omega \times [0,T])$, i.e., the source term is square-integrable in both time and space.

4.Initial condition: $f_0(\mathbf{u}) \in L^2(\Omega)$, i.e., the initial distribution function is squareintegrable.

5.Boundary condition: We consider homogeneous Dirichlet boundary conditions $f|_{\partial\Omega} = 0$. This means the particle number density is zero on the boundary.

Function spaces:

We expect to find solutions f in the Sobolev space $L^2([0,T]; H_0^1(\Omega))$, with time derivative $\frac{\partial f}{\partial t}$ in the dual space $L^2([0,T]; H^{-1}(\Omega))$.

Derivation of Weak Formulation

To derive the weak formulation of the equation, multiply both sides by a smooth test function $\varphi(\mathbf{u}) \in C_0^\infty(\Omega)$ (smooth functions vanishing on the boundary of Ω) and integrate over Ω. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, the final weak form holds for all $\varphi \in H_0^1(\Omega)$.[34]

$$\int_\Omega \frac{\partial f}{\partial t}\varphi d\mathbf{u} + \int_\Omega v^i \frac{\partial f}{\partial x^i}\varphi d\mathbf{u} + \int_\Omega \mathcal{F}^i \frac{\partial f}{\partial p^i}\varphi d\mathbf{u} = \int_\Omega \frac{\partial}{\partial p^i}\left(K^{ij}\frac{\partial f}{\partial p^j}\right)\varphi d\mathbf{u} + \int_\Omega S\varphi d\mathbf{u} \quad (5)$$

Now, perform integration by parts on each term:

1.Time term: $\int_\Omega \frac{\partial f}{\partial t}\varphi d\mathbf{u} = \left\langle \frac{\partial f}{\partial t}, \varphi \right\rangle_{H^{-1},H_0^1}$, where $\langle \cdot,\cdot \rangle$ denotes the duality pairing between H^{-1} and H_0^1.

2.Spatial convection term: $\int_\Omega v^i \frac{\partial f}{\partial x^i}\varphi d\mathbf{u}$. Using integration by parts $\int (\partial_i A) B d\mathbf{u} = -\int A(\partial_i B) d\mathbf{u} +$ boundary term.

$$\int_\Omega v^i \frac{\partial f}{\partial x^i}\varphi d\mathbf{u} = -\int_\Omega f \frac{\partial}{\partial x^i}\left(v^i\varphi\right) d\mathbf{u} \quad (6)$$

Since $\varphi \in H_0^1(\Omega)$, $\varphi = 0$ on $\partial\Omega$, so the boundary term is zero.

Expanding $\frac{\partial}{\partial x^i}\left(v^i\varphi\right) = v^i \frac{\partial \varphi}{\partial x^i} + \varphi \frac{\partial v^i}{\partial x^i}$.

Since $v^i = \frac{\partial^i}{\partial t}$ depends only on momentum $\mathbf{p}$, not on position $\mathbf{x}$, $\frac{\partial v^i}{\partial x^i} = 0$.

Thus, $\int_\Omega v^i \frac{\partial f}{\partial x^i} \varphi d\mathbf{u} = -\int_\Omega f v^i \frac{\partial \varphi}{\partial x^i} d\mathbf{u}$.

3.Momentum convection term: $\int_\Omega \mathcal{F}^i \frac{\partial f}{\partial p^i} \varphi d\mathbf{u}$. Similarly, integration by parts:

$$\int_\Omega \mathcal{F}^i \frac{\partial f}{\partial p^i} \varphi d\mathbf{u} = -\int_\Omega f \frac{\partial}{\partial p^i}\left(\mathcal{F}^i \varphi\right) d\mathbf{u} \quad (7)$$

Boundary term is zero because $\varphi|_{\partial\Omega} = 0$.

Expanding $\frac{\partial}{\partial p^i}\left(\mathcal{F}^i \varphi\right) = \mathcal{F}^i \frac{\partial \varphi}{\partial p^i} + \varphi \frac{\partial \mathcal{F}^i}{\partial p^i}$.

Thus, $\int_\Omega \mathcal{F}^i \frac{\partial f}{\partial p^i} \varphi d\mathbf{u} = -\int_\Omega f \mathcal{F}^i \frac{\partial \varphi}{\partial p^i} d\mathbf{u} - \int_\Omega f \frac{\partial \mathcal{F}^i}{\partial p^i} \varphi d\mathbf{u}.$.

4.Diffusion term: $\int_\Omega \frac{\partial}{\partial p^i}\left(K^{ij} \frac{\partial f}{\partial p^j}\right) \varphi d\mathbf{u}$. Integration by parts:

$$\int_\Omega \frac{\partial}{\partial p^i}\left(K^{ij} \frac{\partial f}{\partial p^j}\right) \varphi d\mathbf{u} = -\int_\Omega K^{ij} \frac{\partial f}{\partial p^j} \frac{\partial \varphi}{\partial p^i} d\mathbf{u} \quad (8)$$

Boundary term is zero because $\varphi|_{\partial\Omega} = 0$. This relies on the flux $K^{ij} \frac{\partial f}{\partial p^j}$ at the boundary.

5.Source term: $\int_\Omega S \varphi d\mathbf{u} = (S, \varphi)_{L^2(\Omega)}$.

Combining all terms, the weak form of the equation is:

$$\left\langle \frac{\partial f}{\partial t}, \varphi \right\rangle + \int_\Omega K^{ij} \frac{\partial f}{\partial p^i} \frac{\partial \varphi}{\partial p^i} d\mathbf{u} - \int_\Omega f v^i \frac{\partial \varphi}{\partial x^i} d\mathbf{u} - \int_\Omega f \mathcal{F}^i \frac{\partial \varphi}{\partial p^i} d\mathbf{u} - \int_\Omega f \frac{\partial \mathcal{F}^i}{\partial p^i} \varphi d\mathbf{u} = \left(S, \varphi\right) \quad (9)$$

for all $\varphi \in H_0^1\left(\Omega\right)$.

For conciseness, we define a bilinear form $a\left(f, \varphi\right)$ and a linear form $L\left(\varphi\right)$:

$$a\left(f, \varphi\right) = \int_\Omega K^{ij} \frac{\partial f}{\partial p^j} \frac{\partial \varphi}{\partial p^i} d\mathbf{u} - \int_\Omega f v^i \frac{\partial \varphi}{\partial x^i} d\mathbf{u} - \int_\Omega f \mathcal{F}^i \frac{\partial \varphi}{\partial p^i} d\mathbf{u} - \int_\Omega f \frac{\partial \mathcal{F}^i}{\partial p^i} \varphi d\mathbf{u} \quad (10)$$

$$L\left(\varphi\right) = \left(S, \varphi\right) \quad \text{(Formula 11)}$$

The weak form can then be written as:

$$\left\langle \frac{\partial f}{\partial t}, \varphi \right\rangle + a\left(f, \varphi\right) = L\left(\varphi\right) \quad \text{for all } \varphi \in H_0^1\left(\Omega\right) \quad (12)$$

with initial condition $f\left(\mathbf{u}, 0\right) = f_0\left(\mathbf{u}\right)$.

4.2 Proof of Existence

The Galerkin method is a powerful tool for constructing a sequence of approximate solutions and proving their convergence to the true solution through a priori estimates[18].

4.2.1 Construction of Galerkin Approximations

1.Choice of basis functions: $H_0^1(\Omega)$ is a separable Hilbert space, so there exists a complete orthogonal basis of functions $\{\phi_k(\mathbf{u})\}_{k=1}^{\infty}$. These basis functions are typically eigenfunctions of the corresponding elliptic operator, such as the Laplace operator.

2.Construction of approximate solutions: We seek a finite-dimensional approximatesolution $f_N(\mathbf{u},t)$, which is a linear combination of the first N basis functions, with coefficients as functions of time t:

$$f_N(\mathbf{u},t)=\sum_{k=1}^{N}c_k(t)\phi_k(\mathbf{u}) \quad \text{(Formula 13)}$$

3.Galerkin equations: Substitute f_N into the weak form and require it to hold for each basis function ϕ_j, $j=1,\ldots,N$. This transforms the partial differential equation into a system of ordinary differential equations:

$$\left\langle\frac{\partial f_N}{\partial t},\phi_j\right\rangle+a\left(f_N,\phi_j\right)=\left(S,\phi_j\right)\quad \text{for } j=1,\ldots,N \quad (14)$$

Due to the orthogonality of the basis functions (if chosen to be L^2 -orthogonal), $\left\langle\frac{\partial f_N}{\partial t},\phi_j\right\rangle=\sum_{k=1}^{N}\frac{dc_k}{dt}(\phi_k,\phi_j)_{L^2}=\frac{dc_j}{dt}$.

Thus, the system becomes:

$$\frac{dc_j}{dt}+\sum_{k=1}^{N}c_k(t)\,a\left(\phi_k,\phi_j\right)=\left(S,\phi_j\right)\quad \text{for } j=1,\ldots,N \quad (15)$$

This is a linear system of ordinary differential equations, with coefficients $a\left(\phi_k,\phi_j\right)$ being constants (since ϕ_k are time-independent), and the right-hand side $\left(S,\phi_j\right)$ time-dependent.

4.Initial conditions: The corresponding initial conditions are $c_j(0)=(f_0,\phi_j)_{L^2(\Omega)}$.

By the theory of ordinary differential equations (e.g., Picard-Lindelöf theorem), for given initial conditions, there exists a unique solution $\mathbf{c}(t)=\left(c_1(t),\ldots,c_N(t)\right)$ on $[0,T]$. Thus, the approximate solution $f_N(\mathbf{u},t)$ exists on $[0,T]$.

4.2.2 A Priori Estimates

A priori estimates are the core of the Galerkin method, proving that the sequence of approximate solutions is uniformly bounded in certain function spaces, allowing extraction of convergent subsequences.

Multiply the j-th Galerkin equation by $c_j(t)$ and sum over j from 1 to N. This is equivalent to choosing the test function $\varphi=f_N$ in the weak form.

$$\frac{1}{2}\frac{d}{dt}\left\|f_N(t)\right\|_{L^2(\Omega)}^2+a\left(f_N,f_N\right)=(S,f_N)_{L^2(\Omega)} \quad (16)$$

Now analyze the bilinear form $a\left(f_N,f_N\right)$:

$$a\left(f_N,f_N\right)=\int_\Omega K^{ij}\frac{\partial f_N}{\partial p^j}\frac{\partial f_N}{\partial p^i}d\mathbf{u}-\int_\Omega f_N v^i\frac{\partial f_N}{\partial x^i}d\mathbf{u}-\int_\Omega f_N\mathcal{F}^i\frac{\partial f_N}{\partial p^i}d\mathbf{u}-\int_\Omega f_N\frac{\partial\mathcal{F}^i}{\partial p^i}f_N d\mathbf{u} \quad (17)$$

Process each term:

1.Diffusion term: $\int_\Omega K^{ij}\frac{\partial f_N}{\partial p^j}\frac{\partial f_N}{\partial p^i}d\mathbf{u}\geq\lambda\left\|\nabla_{\mathbf{p}}f_N\right\|_{L^2(\Omega)}^2$ (by positive definiteness of K^{ij}).

2.Spatial convection term: $-\int_{\Omega} f_N v^i \frac{\partial f_N}{\partial x^i} d\mathbf{u} = -\frac{1}{2}\int_{\Omega} v^i \frac{\partial \left(f_N^2\right)}{\partial x^i} d\mathbf{u}$. By integration by parts, using $\frac{\partial v^i}{\partial x^i} = 0$ and $f_N \mid_{\partial\Omega} = 0$ (boundary term zero), this term is $\frac{1}{2}\int_{\Omega} f_N^2 \frac{\partial v^i}{\partial x^i} d\mathbf{u} = 0$.

3.Momentum convection term: $-\int_{\Omega} f_N \mathcal{F}^i \frac{\partial f_N}{\partial p^i} d\mathbf{u} = -\frac{1}{2}\int_{\Omega} \mathcal{F}^i \frac{\partial \left(f_N^i\right)}{\partial p^i} d\mathbf{u}$. By integration by parts, with $f_N \mid_{\partial\Omega} = 0$ (boundary term zero), this becomes $\frac{1}{2}\int_{\Omega} f_N^2 \frac{\partial \mathcal{F}^i}{\partial p^i} d\mathbf{u}$.

4.Force field divergence term: $-\int_{\Omega} f_N \frac{\partial \mathcal{F}^i}{\partial p^i} f_N d\mathbf{u} = -\int_{\Omega} f_N^2 \frac{\partial \mathcal{F}^i}{\partial p^i} d\mathbf{u}$.

Combining the last three terms: $-\frac{1}{2}\int_{\Omega} f_N^2 \frac{\partial \mathcal{F}^i}{\partial p^i} d\mathbf{u} - \int_{\Omega} f_N^2 \frac{\partial \mathcal{F}^i}{\partial p^i} d\mathbf{u} = -\frac{3}{2}\int_{\Omega} f_N^2 \frac{\partial \mathcal{F}^i}{\partial p^i} d\mathbf{u}$. Wait, correcting the combination from original: actually from term 3 and 4: $\frac{1}{2}\int f_N^2\, \partial \mathcal{F}^i / \partial p^i - \int f_N^2\, \partial \mathcal{F}^i / \partial p^i = -\frac{1}{2}\int f_N^2\, \partial \mathcal{F}^i / \partial p^i$.

Let $C_F = \sup_{\mathbf{u}\in\Omega} \left| \frac{1}{2} \frac{\partial \mathcal{F}^i}{\partial p^i} \right|$, then $-\frac{1}{2}\int_{\Omega} f_N^2 \frac{\partial \mathcal{F}^i}{\partial p^i} d\mathbf{u} \geq -C_F \left\| f_N \right\|_{L^2(\Omega)}^2$.

So, $a\left(f_N, f_N\right) \geq \lambda \left\| \nabla_{\mathbf{p}} f_N \right\|_{L^2(\Omega)}^2 - C_F \left\| f_N \right\|_{L^2(\Omega)}^2$.

Substitute back into the energy equation:

$$\frac{1}{2}\frac{d}{dt} \| f_N\left(t\right) \|_{L^2(\Omega)}^2 + \lambda \| \nabla_{\mathbf{p}} f_N \|_{L^2(\Omega)}^2 - C_F \| f_N \|_{L^2(\Omega)}^2 \leq \| S\left(t\right) \|_{L^2(\Omega)} \| f_N\left(t\right) \|_{L^2(\Omega)} \quad (18)$$

Using Young's inequality $ab \leq \frac{1}{2\epsilon}\alpha^2 + \frac{\epsilon}{2} b^2$, with $\epsilon = 1$:

$$\left\| S\left(t\right) \right\|_{L^2(\Omega)} \left\| f_N\left(t\right) \right\|_{L^2(\Omega)} \leq \frac{1}{2}\left\| S\left(t\right) \right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\| f_N\left(t\right) \right\|_{L^2(\Omega)}^2 \quad (19)$$

Substitute and rearrange:

$$\frac{1}{2}\frac{d}{dt}\left\| f_N\left(t\right) \right\|_{L^2(\Omega)}^2 + \lambda \left\| \nabla_{\mathbf{p}} f_N \right\|_{L^2(\Omega)}^2 \leq \left(C_F + \frac{1}{2} \right) \left\| f_N\left(t\right) \right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\| S\left(t\right) \right\|_{L^2(\Omega)}^2 \quad (20)$$

Ignoring the non-negative momentum gradient term and multiplying by 2:

$$\frac{d}{dt}\left\| f_N\left(t\right) \right\|_{L^2(\Omega)}^2 \leq \left(2C_F + 1\right) \left\| f_N\left(t\right) \right\|_{L^2(\Omega)}^2 + \left\| S\left(t\right) \right\|_{L^2(\Omega)}^2 \quad (21)$$

Let $C_{GR} = 2C_F + 1$. Applying Grönwall's inequality:

$$\left\| f_N\left(t\right) \right\|_{L^2(\Omega)}^2 \leq \left\| f_N\left(0\right) \right\|_{L^2(\Omega)}^2 e^{C_{GR} t} + \int_0^t e^{C_{GR}(t-\tau)} \left\| S\left(\tau\right) \right\|_{L^2(\Omega)}^2 d\tau \quad (22)$$

Since $\left\| f_N\left(0\right) \right\|_{L^2(\Omega)}^2 \leq \left\| f_0 \right\|_{L^2(\Omega)}^2$ (as $f_N\left(0\right)$ is the L^2 projection of f_0) and $S \in L^2\left(\Omega \times \left[0, T\right]\right)$, the integral term on the right is bounded on $\left[0, T\right]$. This indicates that $\left\| f_N\left(t\right) \right\|_{L^2(\Omega)}^2$ is uniformly bounded on the entire time interval $\left[0, T\right]$, with the bound independent of N.

Next, integrate the inequality $\frac{1}{2}\frac{d}{dt}\left\|f_N(t)\right\|_{L^2(\Omega)}^2+\lambda\left\|\nabla_{\mathbf{p}}f_N\right\|_{L^2(\Omega)}^2\leq\left(C_F+\frac{1}{2}\right)\frac{1}{2}\left\|S(t)\right\|_{L^2(\Omega)}^2$ from 0 to T:

$$\begin{gathered}\frac{1}{2}\left\|f_N(T)\right\|_{L^2(\Omega)}^2-\frac{1}{2}\left\|f_N(0)\right\|_{L^2(\Omega)}^2+\lambda\int_0^T\left\|\nabla_{\mathbf{p}}f_N(t)\right\|_{L^2(\Omega)}^2dt\leq\\ \left(C_F+\frac{1}{2}\right)\int_0^T\left\|f_N(t)\right\|_{L^2(\Omega)}^2dt+\frac{1}{2}\int_0^T\left\|S(t)\right\|_{L^2(\Omega)}^2dt\end{gathered}\quad\text{(Formula 23)}$$

Since $\left\|f_N(t)\right\|_{L^2(\Omega)}^2 z$ is bounded on $[0,T]$, all terms on the right are bounded. Therefore, $\int_0^T\left\|\nabla_{\mathbf{p}}f_N(t)\right\|_{L^2(\Omega)}^2dt$ is also uniformly bounded.

This proves that the sequence $\{f_N\}$ is bounded in $L^\infty([0,T];L^2(\Omega))$;$L^2(\Omega)$) and in $L^2([0,T];H_{\mathbf{p}}^1(\Omega))$ (H^1 norm in the momentum direction).

4.2.3 Convergence

From the a priori estimates, the sequence $\{f_N\}$:

Is uniformly bounded in $L^\infty([0,T];L^2(\Omega))$. Is uniformly bounded in $L^2([0,T];H_{\mathbf{p}}^1(\Omega))$.

Additionally, from the weak form of the Galerkin equations $\left\langle\frac{\partial f_N}{\partial t},\phi\right\rangle=(S,\phi)-a(f_N,\phi)$, using the boundedness of f_N and $\nabla_{\mathbf{p}}f_N$, and the boundedness of coefficients v^i, $\mathcal{F}^i$, K^{ij} can be shown that $\left\|\frac{\partial f_N}{\partial t}\right\|_{L^2([0,T];H^{-1}(\Omega))}$ is also uniformly bounded.

By the Banach-Alaoglu theorem and Aubin-Lions lemma, from the sequence $\{f_N\}$, we can extract a subsequence (still denoted $\{f_N\}$ for simplicity) and a function $f(\mathbf{u},t)$, such that as $N\to\infty$ [18]:

- $f_N\rightharpoonup f$ weakly in $L^2([0,T];H_0^1(\Omega))$.

- $f_N\rightharpoonup f$ weak* in $L^\infty([0,T];L^2(\Omega))$.

- $\frac{\partial f_N}{\partial t}\rightharpoonup\frac{\partial f}{\partial t}$ weakly in $L^2([0,T];H^{-1}(\Omega))$.

-The Aubin-Lions lemma further ensures $f_N\to f$ strongly in $L^2([0,T];L^2(\Omega))$. This strong convergence simplifies the proof since the equation is linear.

4.2.4 Verification that the Limit is a Weak Solution

To prove that the limit function f is a weak solution of the equation, integrate the Galerkin equations over time from 0 to T:

$$\int_0^T\left\langle\frac{\partial f_N}{\partial t},\varphi\right\rangle dt+\int_0^T a(f_N,\varphi)dt=\int_0^T(S,\varphi)dt\quad(24)$$

for any $\varphi\in H_0^1(\Omega)$.

As $N\to\infty$, due to the linearity and continuity under weak convergence, we can interchange the limit with the integrals and weak forms.

The left first term: $\int_0^T\left\langle\frac{\partial f_N}{\partial t},\varphi\right\rangle dt\to\int_0^T\left\langle\frac{\partial f}{\partial t},\varphi\right\rangle dt$.

The left second term: $\int_0^T a\left(f_N,\varphi\right)dt \to \int_0^T a\left(f,\varphi\right)dt$.

The right term: $\int_0^T (S,\varphi)dt$ remains unchanged.

Thus, the limit function f satisfies:

$$\int_0^T\left[\left\langle \frac{\partial f}{\partial t},\varphi\right\rangle + a\left(f,\varphi\right)\right]dt = \int_0^T (S,\varphi)dt \quad (25)$$

for all $\phi \in H_0^1\left(\Omega\right)$, meaning f is a weak solution of the equation.

Finally, the initial condition $f\left(\mathbf{u},0\right) = f_0\left(\mathbf{u}\right)$ is typically satisfied using the property $f \in C\left(\left[0,T\right];L^2\left(\Omega\right)\right)$, which can be proved by stronger results from the Aubin-Lions lemma or energy estimates.

4.3 Proof of Uniqueness

Uniqueness of solutions is typically proved by comparing two assumed solutions and using energy methods[34]. This approach is common for parabolic PDEs[35].

Assume there are two weak solutions f_1 and f_2 satisfying the same initial condition $f\left(\mathbf{u},0\right) = f_0\left(\mathbf{u}\right)$ and the same source term $S\left(\mathbf{u},t\right)$.

Let $w = f_1 - f_2$. Then w satisfies the homogeneous equation:

$$\left\langle \frac{\partial w}{\partial t},\varphi\right\rangle + a\left(w,\varphi\right) = 0 \quad \text{for all } \varphi \in H_0^1\left(\Omega\right) \quad (26)$$

with $w\left(\mathbf{u},0\right) = f_1\left(\mathbf{u},0\right) - f_2\left(\mathbf{u},0\right) = 0$.

Choose the test function $\varphi = w$. $w \in L^2\left(\left[0,T\right];H_0^1\left(\Omega\right)\right)$, this choice is permissible.

$$\frac{1}{2}\frac{d}{dt}\left\|w\left(t\right)\right\|_{L^2(\Omega)}^2 + a\left(w,w\right) = 0 \quad (27)$$

From the analysis in the existence proof, $a\left(w,w\right) \ge \lambda\left\|\nabla_{\mathbf{p}} w\right\|_{L^2(\Omega)}^2 - C_F\left\|w\right\|_{L^2(\Omega)}^2$. Substitute and arrange:

$$\frac{1}{2}\frac{d}{dt}\left\|w\left(t\right)\right\|_{L^2(\Omega)}^2 + \lambda\left\|\nabla_{\mathbf{p}} w\right\|_{L^2(\Omega)}^2 - C_F\left\|w\right\|_{L^2(\Omega)}^2 \le 0 \quad (28)$$

Since $\lambda\left\|\nabla_{\mathbf{p}} w\right\|_{L^2(\Omega)}^2 \ge 0$, we obtain a differential inequality:

$$\frac{1}{2}\frac{d}{dt}\left\|w\left(t\right)\right\|_{L^2(\Omega)}^2 \le C_F\left\|w\left(t\right)\right\|_{L^2(\Omega)}^2 \quad (29)$$

or:

$$\frac{d}{dt}\left\|w\left(t\right)\right\|_{L^2(\Omega)}^2 \le 2C_F\left\|w\left(t\right)\right\|_{L^2(\Omega)}^2 \quad (30)$$

Applying Grönwall's inequality again:

$$\left\|w\left(t\right)\right\|_{L^2(\Omega)}^2 \le \left\|w\left(0\right)\right\|_{L^2(\Omega)}^2 e^{2C_F t} \quad (31)$$

Since $\left\|w\left(0\right)\right\|_{L^2(\Omega)}^2 = 0$, we deduce $\left\|w\left(t\right)\right\|_{L^2(\Omega)}^2 \le 0$.

Because the norm must be non-negative, $\|w(t)\|_{L^2(\Omega)}^2 = 0$ for all $t \in [0,T]$.

This means $w(\mathbf{u},t) = 0$ almost everywhere, hence $f_1 = f_2$. Therefore, the weak solution of the equation is unique.

4.4 Stability of Solutions

Stability refers to the continuous dependence of the solution on small perturbations in initial conditions and source terms. If small changes in initial conditions or source terms result in small changes in the solution, the equation is stable[34].

Consider two solutions f_1 and f_2, corresponding to initial conditions $f_{0,1}$, $f_{0,2}$ and source terms S_1, S_2.

Let $w = f_1 - f_2$. Then w satisfies the non-homogeneous equation:

$$\left\langle \frac{\partial w}{\partial t}, \varphi \right\rangle + a(w,\varphi) = (S_1 - S_2, \varphi) \quad (32)$$

with $w(\mathbf{u},0) = f_{0,1}(\mathbf{u}) - f_{0,2}(\mathbf{u})$.

Choose the test function $\varphi = w$:

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2(\Omega)}^2 + a(w,w) = (S_1 - S_2, w)_{L^2(\Omega)} \quad (33)$$

Using $a(w,w) \ge \lambda \|\nabla_{\mathbf{p}} w\|_{L^2(\Omega)}^2 - C_F \|w\|_{L^2(\Omega)}^2$ and Young's inequality $(S_1 - S_2, w) \le \|S_1 - S_2\|_{L^2} \|w\|_{L^2} \le \frac{1}{2}\|S_1 - S_2\|_{L^2}^2 + \frac{1}{2}\|w\|_{L^2}^2$:

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2(\Omega)}^2 \le C_F \|w(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|S_1(t) - S_2(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|w(t)\|_{L^2(\Omega)}^2 \quad (34)$$

$$\frac{d}{dt}\|w(t)\|_{L^2(\Omega)}^2 \le (2C_F + 1)\|w(t)\|_{L^2(\Omega)}^2 + \|S_1(t) - S_2(t)\|_{L^2(\Omega)}^2 \quad (35)$$

Let $C_{stab} = 2C_F + 1$. Applying Grönwall's inequality again:

$$\|w(t)\|_{L^2(\Omega)}^2 \le \|w(0)\|_{L^2(\Omega)}^2 e^{C_{stab}t} + \int_0^t e^{C_{stab}(t-\tau)} \|S_1(\tau) - S_2(\tau)\|_{L^2(\Omega)}^2 d\tau \quad (36)$$

This inequality quantifies how perturbations propagate over time. It shows that the difference between solutions $\|f_1(t) - f_2(t)\|_{L^2(\Omega)}$ grows exponentially with the differences in initial conditions $\|f_{0,1} - f_{0,2}\|_{L^2(\Omega)}$ and source perturbations $\|S_1 - S_2\|_{L^2([0,T];L^2(\Omega))}$. Thus, if the initial perturbation and source perturbation are sufficiently small, the difference between solutions remains small. Therefore, the solution of the equation is continuously dependent on the initial conditions and source terms in the L^2 norm sense, i.e., it is stable [34,35].

5. Non-Negativity of Solutions

In physics, the distribution function $f(\mathbf{u},t)$ represents particle number density, so it must be non-negative ($f \ge 0$). This is an important physical constraint[21]. We prove that, under appropriate initial conditions and source terms, the solution of the Fokker-Planck equation maintains non-negativity.

Assumptions:

1.Initial condition: $f_0(\mathbf{u}) \ge 0$ almost everywhere (i.e.) in Ω.

2.Source term: $S(\mathbf{u},t) \ge 0$ almost everywhere in $\Omega \times [0,T]$.

3.Boundary condition: Homogeneous Dirichlet boundary condition $f|_{\partial\Omega}=0$.

4.Diffusion coefficient: K^{ij} is symmetric and positive definite, i.e., $K^{ij}\xi^i\xi^j \ge \lambda|\xi|^2$ for some positive constant $\lambda>0$.

Proof:

We use a common technique to prove non-negativity by introducing an auxiliary function to measure the negative part of the solution. Define $f_-(\mathbf{u},t)=\max(-f(\mathbf{u},t),0)$. Our goal is to prove $f_-\equiv 0$, meaning $f\ge 0$.

Choose the test function $\varphi=-f_-$. This choice is allowed because if $f\in H_0^1(\Omega)$, then its negative part f_- also belongs to $H_0^1(\Omega)$.

Substitute $\varphi=-f_-$ into the weak form:

$$\left\langle \frac{\partial f}{\partial t},-f_-\right\rangle + a(f,-f_-)=(S,-f_-) \quad (37)$$

Note that the integral $\int_\Omega(\cdot)_- d\mathbf{u}$ is only over the region− $\{\mathbf{u}\in\Omega: f(\mathbf{u},t)<0\}$, because $f_-=0$ where $f\ge 0$. In this negative region, $-f_-=f$.

Analyze each term:

1.Time term: $\left\langle \frac{\partial f}{\partial t},-f_-\right\rangle=\int_{\{f<0\}}\frac{\partial f}{\partial t} f d\mathbf{u}=\frac{1}{2}\frac{d}{dt}\int_{\{f<0\}} f^2 d\mathbf{u}=\frac{1}{2}\frac{d}{dt}\left\|f_-(t)\right\|^2_{L^2(\Omega)}$.

2.Bilinear form $a(f,-f_-)$ terms:

(*a*).Spatial convection term: $-\int_\Omega f v^i \frac{\partial(-f_-)}{\partial x^i} d\mathbf{u}=-\int_{\{f<0\}} f v^i \frac{\partial f}{\partial x^i} d\mathbf{u}$..

As in the existence proof, this term is $-\frac{1}{2}\int_{\{f<0\}} v^i \frac{\partial f^2}{\partial x^i} d\mathbf{u}$. Integrating by parts over $\{f<0\}$, using $\frac{\partial f^i}{\partial x^i}=0$ and $f|_{\partial\Omega}=0$ (implying $f_-|_{\partial\Omega}=0$), this term is 0.

(*b*).Momentum convection term: $-\int_\Omega f\mathcal{F}^i \frac{\partial(-f_-)}{\partial p^i} du=-\int_{\{f<0\}} f\mathcal{F}^i \frac{\partial f}{\partial p^i} du$. This is $-\frac{1}{2}\int_{\{f<0\}}\mathcal{F}^i\frac{\partial f^2}{\partial p^i} d\mathbf{u}$. Integrating by parts over $\{f<0\}$, this becomes $\frac{1}{2}\int_{\{f<0\}} f^2\frac{\partial\mathcal{F}^i}{\partial p^i} d\mathbf{u}$.

(*c*).Force field divergence term: $-\int_\Omega f\frac{\partial\mathcal{F}^i}{\partial p^i}(-f_-) d\mathbf{u}=-\int_{\{f<0\}} f^2\frac{\partial\mathcal{F}^i}{\partial p^i} d\mathbf{u}$.

Combining these three terms: $\left(\frac{1}{2}-1\right)\int_{\{f<0\}} f^2\frac{\partial\mathcal{F}^i}{\partial p^i} d\mathbf{u}=-\frac{1}{2}\int_{\{f<0\}} f^2\frac{\partial\mathcal{F}^i}{\partial p^i} d\mathbf{u}$.

Using $C_F=\sup_{\mathbf{u}\in\Omega}\left|\frac{1}{2}\frac{\partial\mathcal{F}^i}{\partial p^i}\right|$, this term $\le C_F\left\|f_-(t)\right\|^2_{L^2(\Omega)}$.

3.Diffusion term: $\int_\Omega K^{ij}\frac{\partial f}{\partial p^i}\frac{\partial(-f_-)}{\partial p^i} d\mathbf{u}=\int_{\{f<0\}} K^{ij}\frac{\partial f}{\partial p^j}\frac{\partial f}{\partial p^i} d\mathbf{u}$.

By positive definiteness of K^{ij} , this term $\ge\lambda\int_{\{f<0\}}|\nabla_{\mathbf{p}} f|^2 d\mathbf{u}=\lambda\left\|\nabla_{\mathbf{p}} f_-\right\|^2_{L^2(\Omega)}$.

4.Source term: $\left(S, -f_-\right) = \int_{\{f<0\}} Sf d\mathbf{u}$.

Since we assume $S \geq 0$ and in the region $\{f < 0\}$, $f < 0$, thus $Sf \leq 0$. So this term is non-positive.

Integrating all terms, we obtain the inequality:

$$\frac{1}{2}\frac{d}{dt}\left\|f_-\left(t\right)\right\|_{L^2(\Omega)}^2 + \lambda\left\|\nabla_{\mathbf{p}} f_-\right\|_{L^2(\Omega)}^2 - C_F\left\|f_-\left(t\right)\right\|_{L^2(\Omega)}^2 \leq \int_{\{f<0\}} Sf d\mathbf{u} \leq 0 \quad (38)$$

Ignoring the non-negative momentum gradient term $\lambda\left\|\nabla_{\mathbf{p}} f_-\right\|_{L^2(\Omega)}^2$, and arranging:

$$\frac{d}{dt}\left\|f_-\left(t\right)\right\|_{L^2(\Omega)}^2 \leq 2C_F\left\|f_-\left(t\right)\right\|_{L^2(\Omega)}^2 \quad (39)$$

Applying Grönwall's inequality:

$$\left\|f_-\left(t\right)\right\|_{L^2(\Omega)}^2 \leq \left\|f_-\left(0\right)\right\|_{L^2(\Omega)}^2 e^{2C_F t} \quad (40)$$

From the initial condition $f_0 \geq 0$, this means $f_-\left(0\right) = \max\left(-f_0, 0\right) = 0$.

Thus, $\left\|f_-\left(t\right)\right\|_{L^2(\Omega)}^2 = 0$ for all $t \in \left[0, T\right]$.

This means $f_-\left(\mathbf{u}, t\right) = 0$ almost everywhere, hence $f\left(\mathbf{u}, t\right) \geq 0$ almost everywhere. This proves the non-negativity of the solution.

6. Derivation of Conservation Laws and Physical Interpretation

Conservation laws describe how physical quantities change during system evolution. We integrate over the entire phase space $\mathbb{R}^6$, assuming the distribution function f and its relevant derivatives decay sufficiently at infinity so that all boundary terms in integration by parts are zero[19]. This approach is fundamental in kinetic theory [31].

6.1 Particle Number Conservation

Physical quantity: Total particle number $N_{tot}\left(t\right) = \int_{\mathbb{R}^6} f\left(\mathbf{x}, \mathbf{p}, t\right) d^3x d^3p$ Integrate the original equation over the entire phase space $\mathbb{R}^6$ [19]:

$$\int_{\mathbb{R}^6}\left(\frac{\partial f}{\partial t} + v^i\frac{\partial f}{\partial x^i} + \mathcal{F}^i\frac{\partial f}{\partial p^i}\right)d\mathbf{u} = \int_{\mathbb{R}^6}\left(\frac{\partial}{\partial p^i}\left(K^{ij}\frac{\partial f}{\partial p^j}\right) + S\right)d\mathbf{u} \quad (41)$$

1. $\int_{\mathbb{R}^6}\frac{\partial f}{\partial t}d\mathbf{u} = \frac{d}{dt}\int_{\mathbb{R}^6} f d\mathbf{u} = \frac{d}{dt}N_{tot}$.

2. $\int_{\mathbb{R}^6} v^i\frac{\partial f}{\partial x^i}d\mathbf{u} = -\int_{\mathbb{R}^6} f\frac{\partial v^i}{\partial x^i}d\mathbf{u} = 0$ (by integration by parts, using v^i independent of x^i, boundary term zero).

3. $\int_{\mathbb{R}^6}\mathcal{F}^i\frac{\partial f}{\partial p^i}d\mathbf{u} = -\int_{\mathbb{R}^6} f\frac{\partial \mathcal{F}^i}{\partial p^i}d\mathbf{u}$ (by integration by parts, boundary term zero).

4. $\int_{\mathbb{R}^6}\frac{\partial}{\partial p^i}\left(K^{ij}\frac{\partial f}{\partial p^j}\right)d\mathbf{u} = 0$ (by integration by parts, boundary flux zero).

5. $\int_{\mathbb{R}^6} Sd\mathbf{u}$.

Summing the terms, the rate of change equation for total particle number is:

$$\frac{dN_{tot}}{dt} - \int_{\mathbb{R}^6} f \frac{\partial \mathcal{F}^i}{\partial p^i} d\mathbf{u} = \int_{\mathbb{R}^6} S d\mathbf{u} \quad (42)$$

$$\frac{dN_{tot}}{dt} = \int_{\mathbb{R}^6} S d\mathbf{u} + \int_{\mathbb{R}^6} f \frac{\partial \mathcal{F}^i}{\partial p^i} d\mathbf{u} \quad (43)$$

Physical interpretation: The total particle number is generally not conserved. Its rate of change depends on two main factors:

-Source-sink term $\int S d\mathbf{u}$: This is the most direct source of particle number change.

If $S > 0$, it indicates new particle generation or injection; if $S < 0$, particle removal or annihilation.

-Force field divergence term $\int f \frac{\partial \mathcal{F}^i}{\partial p^i} d\mathbf{u}$: This reflects the compression or expansion effect of the external force field in momentum space. If $\frac{\partial \mathcal{F}^i}{\partial p^i} \neq 0$, the force field is not divergencefree in momentum space. For example, for the relativistic Lorentz force $q\epsilon^{ijk} v^j B^k$, since $v^j = p^j / (\gamma m)$ and γ depends on p , $\frac{\partial}{\partial p^i}\left(q\epsilon^{ijk} v^j B^k\right)$ is generally nonzero, leading to nonconservation of particle number. This is a more complex mechanism for particle number change, related to ”squeezing” in phase space.

The total particle number is conserved only when the source-sink term is zero and the total force $\mathcal{F}^i$ is divergence-free in momentum space (i.e., $\frac{\partial \mathcal{F}^i}{\partial p^i} = 0$)

6.2 Total Momentum Conservation

Physical quantity: Total momentum $P_{tot}^k(t) = \int_{\mathbb{R}^6} p^k f(\mathbf{x}, \mathbf{p}, t) d^3x d^3p$ [19,31]

Multiply the original equation by the momentum component p^k and integrate over $\mathbb{R}^6$:

$$\int_{\mathbb{R}^6} p^k \left(\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} + \mathcal{F}^i \frac{\partial f}{\partial p^i} \right) d\mathbf{u} = \int_{\mathbb{R}^6} p^k \left(\frac{\partial}{\partial p^i}\left(K^{ij} \frac{\partial f}{\partial p^j} \right) + S \right) d\mathbf{u} \quad (44)$$

Analyze each term:

1. $\int_{\mathbb{R}^6} p^k \frac{\partial f}{\partial t} d\mathbf{u} = \frac{d}{dt} P_{tot}^k$.

2. $\int_{\mathbb{R}^6} p^k v^i \frac{\partial f}{\partial x^i} d\mathbf{u} = -\int_{\mathbb{R}^6} f \frac{\partial}{\partial x^i}\left(p^k v^i\right) d\mathbf{u} = 0$ (integration by parts, $p^k v^i$ independent of x^i , boundary term zero).

3. $\int_{\mathbb{R}^6} p^k \mathcal{F}^i \frac{\partial f}{\partial p^i} d\mathbf{u} = -\int_{\mathbb{R}^6} f \frac{\partial}{\partial p^i}\left(p^k \mathcal{F}^i\right) d\mathbf{u}$.

Expanding the integrand: $\frac{\partial}{\partial p^i}\left(p^k \mathcal{F}^i\right) = \frac{\partial p^k}{\partial p^i} \mathcal{F}^i + p^k \frac{\partial \mathcal{F}^i}{\partial p^i} = \delta_{ki} \mathcal{F}^i + p^k \frac{\partial \mathcal{F}^i}{\partial p^i} = \mathcal{F}^k + p^k \frac{\partial \mathcal{F}^i}{\partial p^i}$.

Thus, this term is $-\int_{\mathbb{R}^6} f \mathcal{F}^k d\mathbf{u} - \int_{\mathbb{R}^6} p^k f \frac{\partial \mathcal{F}^i}{\partial p^i} d\mathbf{u}$.

4. $\int_{\mathbb{R}^6} p^k \frac{\partial}{\partial p^i}\left(K^{ij}\frac{\partial f}{\partial p^j}\right) d\mathbf{u} = -\int_{\mathbb{R}^6} K^{ij}\frac{\partial f}{\partial p^j}\frac{\partial p^i}{\partial p^i} d\mathbf{u}$..

Since $\frac{\partial p^k}{\partial p^i} = \delta_{ki}$, this becomes $-\int_{\mathbb{R}^6} K^{kj}\frac{\partial f}{\partial p^j} d\mathbf{u}$.

5. $\int_{\mathbb{R}^6} p^k S d\mathbf{u}$.

Summing the terms, the rate of change equation for total momentum is:

$$\frac{dP^k_{tot}}{dt} = \int_{\mathbb{R}^6} f\mathcal{F}^k d\mathbf{u} + \int_{\mathbb{R}^6} p^k f \frac{\partial \mathcal{F}^i}{\partial p^i} d\mathbf{u} - \int_{\mathbb{R}^6} K^{kj}\frac{\partial f}{\partial p^j} d\mathbf{u} + \int_{\mathbb{R}^6} p^k S d\mathbf{u} \quad (45)$$

Physical interpretation: The total momentum is generally not conserved. Its rate of change is influenced by:

-Total external force R $\int f\mathcal{F}^k d\mathbf{u}$: This is the most direct source of momentum change, representing the macroscopic average of deterministic forces on all particles in the system.

-Force field momentum divergence term $\int p^k f \frac{\partial \mathcal{F}^i}{\partial p^i} d\mathbf{u}$: Similar to particle number conservation, if the force field is not divergence-free in momentum space, it causes additional momentum change.

-Diffusion/friction term $\int K^{kj}\frac{\partial f}{\partial p^j} d\mathbf{u}$: This is momentum dissipation or exchange due to random processes (diffusion and friction) in momentum space. For example, collisions with background medium can transfer particle momentum to the medium, leading to nonconservation of system total momentum.

-Source term $\int p^k S d\mathbf{u}$: Newly generated particles carry momentum, directly changing the system's total momentum.

Total momentum is conserved only when the sum of all non-conserving terms (external force, force field divergence, diffusion/friction, and source) is zero.

6.3 Total Energy Conservation

Physical quantity: Total energy $E_{tot}(t) = \int_{\mathbb{R}^6} E(\mathbf{p}) f(\mathbf{x},\mathbf{p},t) d^3x d^3p$, where $E(\mathbf{p}) = \gamma mc^2 = \sqrt{(|\mathbf{p}|c)^2 + (mc^2)^2}$ is the relativistic total energy (including rest mass energy)[25,26]. Multiply the original equation by energy E and integrate over $\mathbb{R}^6$:

$$\int_{\mathbb{R}^6} E\left(\frac{\partial f}{\partial t} + v^i\frac{\partial f}{\partial x^i} + \mathcal{F}^i\frac{\partial f}{\partial p^i}\right) d\mathbf{u} = \int_{\mathbb{R}^6} E\left(\frac{\partial}{\partial p^i}\left(K^{ij}\frac{\partial f}{\partial p^j}\right) + S\right) d\mathbf{u} \quad (46)$$

Analyze each term:

1. $\int_{\mathbb{R}^6} E\frac{\partial f}{\partial t} d\mathbf{u} = \frac{d}{dt}E_{tot}$.

2. $\int_{\mathbb{R}^6} Ev^i\frac{\partial f}{\partial x^i} d\mathbf{u} = -\int_{\mathbb{R}^6} f\frac{\partial}{\partial x^i}\left(Ev^i\right) d\mathbf{u} = 0$ (integration by parts, Ev^i independent of x^i boundary term zero).

3. $\int_{\mathbb{R}^6} E\mathcal{F}^i\frac{\partial f}{\partial p^i} d\mathbf{u} = -\int_{\mathbb{R}^6} f\frac{\partial}{\partial p^i}\left(E\mathcal{F}^i\right) d\mathbf{u}$.

Expanding the integrand: $\frac{\partial}{\partial p^i}\left(E\mathcal{F}^i\right) = \frac{\partial E}{\partial p^i}\mathcal{F}^i + E\frac{\partial \mathcal{F}^i}{\partial p^i}$

We know $\frac{\partial E}{\partial p^i}=\frac{\partial}{\partial p^i}\left(\sqrt{(|\mathbf{p}|c)^2+(mc^2)^2}\right)=\frac{p^i c^2}{E}=v^i$.

Thus, this term is $-\int_{\mathbb{R}^6} f\left(v^i\mathcal{F}^i\right)d\mathbf{u}-\int_{\mathbb{R}^6} Ef\frac{\partial\mathcal{F}^i}{\partial p^i}d\mathbf{u}$. Note that the magnetic force $q\epsilon^{ijk}v^jB^k$ does no work, i.e., $\left(q\epsilon^{ijk}v^jB^k\right)v^i=q\left(\mathbf{v}\times\mathbf{B}\right)\cdot\mathbf{v}=0$.

4. $\int_{\mathbb{R}^6} E\frac{\partial}{\partial p^i}\left(K^{ij}\frac{\partial f}{\partial p^j}\right)d\mathbf{u}=-\int_{\mathbb{R}^6} K^{ij}\frac{\partial f}{\partial p^j}\frac{\partial f}{\partial p^i}d\mathbf{u}$.

Using $\frac{\partial E}{\partial p^i}=v^i$, this becomes $-\int_{\mathbb{R}^6} K^{ij}\frac{\partial f}{\partial p^j}v^i d\mathbf{u}$.

5. $\int_{\mathbb{R}^6} ESd\mathbf{u}$.

Summing the terms, the rate of change equation for total energy is:

$$\frac{dE_{tot}}{dt}=\int_{\mathbb{R}^6} f\left(\mathbf{v}\cdot\mathbf{F}_{GR}+q\mathbf{v}\cdot\mathbf{E}\right)d\mathbf{u}+\int_{\mathbb{R}^6} Ef\frac{\partial\mathcal{F}^i}{\partial p^i}d\mathbf{u}-\int_{\mathbb{R}^6} K^{ij}\frac{\partial f}{\partial p^j}v^i d\mathbf{u}+\int_{\mathbb{R}^6} ESd\mathbf{u} \quad (47)$$

Physical interpretation: The total energy is generally not conserved. Its rate of change is influenced by:

-External force work term R $\int f\left(\mathbf{v}\cdot\mathbf{F}_{GR}+q\mathbf{v}\cdot\mathbf{E}\right)d\mathbf{u}$: Gravitational and electric fields do work on particles, directly changing particle energy. Note that the Lorentz magnetic force does no work, so it does not contribute directly to this term.

-Force field momentum divergence term $\int Ef\frac{\partial\mathcal{F}^i}{\partial p^i}d\mathbf{u}$: Additional contribution to total energy from nonuniformity or momentum dependence of the force field in momentum space. - Diffusion/friction dissipation term $-\int K^{ij}\frac{\partial f}{\partial p^i}v^i d\mathbf{u}$: Energy dissipation or exchange due to random processes (diffusion and friction) in momentum space. For example, collisions with background medium convert particle kinetic energy to internal energy (heat) of the medium, leading to system energy loss.

-Source term $\int ESd\mathbf{u}$: Newly generated particles carry energy, directly changing the system's total energy.

Total energy is conserved only when the sum of all non-conserving terms is zero.

7. Distribution in Equilibrium States

In physical systems, after long-term evolution, if external conditions remain unchanged, the system often reaches a stable state, namely the equilibrium state. In the equilibrium state, the system's macroscopic properties no longer change with time. For the distribution function $f\left(\mathbf{x},\mathbf{p},t\right)$, this means:

1.Time independence: $\frac{\partial f}{\partial t}=0$.

2.No source-sink term: $S\left(\mathbf{x},\mathbf{p},t\right)=0$.

Let the equilibrium distribution function be $f_0\left(\mathbf{x},\mathbf{p}\right)$. Substituting these conditions into the original Fokker-Planck equation yields the equilibrium equation:

$$v^i\frac{\partial f_0}{\partial x^i}+\mathcal{F}^i\frac{\partial f_0}{\partial p^i}=\frac{\partial}{\partial p^i}\left(K^{ij}\frac{\partial f_0}{\partial p^j}\right) \quad (48)$$

In the context of thermodynamic equilibrium, an important concept is detailed balance. Detailed balance is a stronger condition than macroscopic balance, requiring that the rates of every microscopic process and its reverse in phase space are equal, leading to zero net particle flux $\mathbf{J}_0$ everywhere in phase space[31]. This principle is crucial for understanding equilibrium in statistical mechanics[20].

The particle flux $\mathbf{J}$ in the Fokker-Planck equation consists of two parts in phase space: - Spatial convection flux: $J_x^i = v^i f$

- Momentum drift-diffusion flux: $J_p^i = \mathcal{F}^i J = K^{ij} \dfrac{\partial f}{\partial p^j}$

Under the detailed balance assumption, we require $\mathbf{J}_0 = 0$, which means:

$$v^i f_0 = 0 \quad \text{for all } i \quad (49)$$

$$\mathcal{F}^i f_0 - K^{ij} \frac{\partial f_0}{\partial p^i} = 0 \quad \text{for all } i \quad (50)$$

Physical discussion:

For the first condition $v^i f_0 = 0$: If f_0 is not the trivial zero distribution, this directly implies $v^i = 0$, i.e., particles are at rest. This clearly does not fit a generally moving particle system. In actual physical scenarios, $v^i f_0 = 0$ is often interpreted as no macroscopic particle flow in phase space in equilibrium, or the distribution function f_0 being uniform in space $\left(\dfrac{\partial f_0}{\partial x^i} = 0\right)$, making the spatial convection term vanish in the equilibrium equation. The core equilibrium condition is zero net flow in momentum space:

$$K^{ij} \frac{\partial f_0}{\partial p^j} = \mathcal{F}^i f_0 \quad (51)$$

This equation is key to equilibrium, describing the precise balance in momentum space between particle drift caused by deterministic force $\mathcal{F}^i$ and momentum diffusion caused by random processes (diffusion coefficient K^{ij}).

Thermodynamic Equilibrium Distribution: Maxwell-Boltzmann Distribution

In many physical systems, if particles are in equilibrium with a heat bath at temperature T, and external forces are conservative, the equilibrium distribution function typically follows the Maxwell-Boltzmann distribution [23,24,31]. This distribution has the exponential form:

$$f_{MB}(\mathbf{x},\mathbf{p}) = C\exp\left(-\frac{E_{tot}(\mathbf{x},\mathbf{p})}{kT}\right) = C\exp\left(-\beta E_{tot}(\mathbf{x},\mathbf{p})\right) \quad (52)$$

where C is the normalization constant, $\beta = \dfrac{1}{kT}$ is the inverse temperature (k is Boltzmann's constant), and $E_{tot}(\mathbf{x},\mathbf{p})$ is the total energy of the particle. This form is well-established in statistical mechanics[32]. For relativistic particles, the total energy usually includes kinetic and potential energy:

$$E_{tot}(\mathbf{x},\mathbf{p}) = \sqrt{(|\mathbf{p}|c)^2 + (mc^2)^2} + U_{GR}(\mathbf{x}) + U_E(\mathbf{x}) \quad (53)$$

Here, $\sqrt{(|\mathbf{p}|c)^2 + (mc^2)^2}$ is the relativistic kinetic energy (including rest mass energy), $U_{GR}(\mathbf{x})$ is the gravitational potential energy, and $U_E(\mathbf{x}) = q\phi_E(\mathbf{x})$ is the electrostatic potential energy (ϕ_E is the electrostatic potential).

To verify if the Maxwell-Boltzmann distribution is a solution to the momentum balance condition in equilibrium, first compute the partial derivative of f_{MB} with respect to momentum p^j :

$$\frac{\partial f_{MB}}{\partial p^j} = C\exp\left(-\beta E_{tot}\right)\left(-\beta \frac{\partial E_{tot}}{\partial p^j}\right) = -\beta \frac{\partial E_{tot}}{\partial p^j} f_{MB} \quad (54)$$

Now substitute this expression into the momentum balance condition $K^{ij} \frac{\partial f_0}{\partial p^j} = \mathcal{F}^i f_0$:

$$K^{ij}\left(-\beta \frac{\partial E_{tot}}{\partial p^j} f_{MB}\right) = \mathcal{F}^i f_{MB} \quad (55)$$

Since f_{MB} is generally nonzero in phase space (except at infinity), divide by f_{MB}:

$$-\beta K^{ij} \frac{\partial E_{tot}}{\partial p^j} = \mathcal{F}^i \quad (56)$$

This is the key condition for the Maxwell-Boltzmann distribution to be an equilibrium solution.

Further analyze $\frac{\partial E_{tot}}{\partial p^j}$. For the relativistic energy $E(\mathbf{p}) = \sqrt{(|\mathbf{p}|c)^2 + (mc^2)^2}$:

$$\frac{\partial E_{tot}}{\partial p^j} = \frac{\partial}{\partial p^j}\left(\sqrt{(|\mathbf{p}|c)^2 + (mc^2)^2}\right) + \frac{\partial U_{GR}}{\partial p^j} + \frac{\partial U_E}{\partial p^j} \quad (57)$$

Since potential energies U_{GR} and U_E depend only on position $\mathbf{x}$, not on momentum $\mathbf{p}$, $\frac{\partial U_{GR}}{\partial p^j} = 0$ and $\frac{\partial U_E}{\partial p^j} = 0$.

Thus, $\frac{\partial E_{tot}}{\partial p^j} = \frac{p^j c^2}{\sqrt{(|\mathbf{p}|c)^2 + (mc^2)^2}} = \frac{p^j c^2}{\gamma mc^2} = \frac{p^j}{\gamma m} = v^j$.

Substitute this result back:

$$\mathcal{F}^i = -\beta K^{ij} v^j \quad (58)$$

Physical interpretation: Generalized Einstein relation

This equation is a generalized Einstein relation. It indicates that in thermodynamic equilibrium, the total deterministic force $\mathcal{F}^i$ acting on the particle must be related to its average velocity v^j and the momentum space diffusion coefficient K^{ij} , inversely proportional to the system temperature T . This condition represents the precise balance between the ”drift” effect produced by deterministic forces and the random diffusion effect.

Now substitute the specific form of $\mathcal{F}^i$:

$$F_{GR}^i + qE^i + q\epsilon^{ijk} v^j B^k = -\beta K^{ij} v^j \quad (59)$$

For the Maxwell-Boltzmann distribution to be a strict equilibrium solution, the above relation must hold everywhere in phase space. This imposes strict constraints on the force fields and diffusion coefficients:

1.Influence of magnetic field (Lorentz force): The Lorentz magnetic force $q\epsilon^{ijk} v^j B^k$ is orthogonal to the particle velocity, so it does no work. In the classical Maxwell-Boltzmann distribution, potential energy typically depends only on position. If $\mathbf{B} \neq 0$, the force term $q\epsilon^{ijk} v^j B^k$ must be exactly canceled by $-\beta K^{ij} v^j$. This usually requires the diffusion coefficient K^{ij} to have a very specific structure (e.g., it may include an anisotropic part related to the magnetic field), or the magnetic field effect is weak enough to be neglected. In more general physical scenarios, the presence of magnetic fields may prevent the system from reaching strict Maxwell-Boltzmann thermodynamic equilibrium, possibly reaching a quasisteady state or equilibrium with macroscopic flows (such as rotation).

2.Conservative forces and classical Einstein relation: If gravitational F_{GR}^i and electric force qE^i are conservative (i.e., derivable from negative gradients of potentials $U_{GR}(\mathbf{x})$ and $U_E(\mathbf{x})$), and no magnetic field (or negligible), the condition simplifies to $F_{GR}^i + qE^i = -\beta K^{ij} v^j$. In classical Fokker-Planck equations, when K^{ij} is isotropic constant

$D\delta^{ij}$, and drift is friction $-Ap^i$, the Einstein relation is usually $A=\beta D$. In our generalized form, this relation is more complex, requiring precise coupling between conservative force terms and momentum space diffusion through temperature.

3.Spatial uniformity or zero spatial flow: Although the spatial convection term $v^i \frac{\partial f_0}{\partial x^i}$ was omitted in the proof, in the equilibrium equation, this term must be zero. This means that in equilibrium, either the distribution function f_0 is uniform in space $\left(\frac{\partial f_0}{\partial x^i}=0\right)$, or there is complex spatial-momentum coupling making $\int v^i f_0 d^3 p=0$ (no macroscopic velocity), or $v^i \frac{\partial f_0}{\partial x^i}$ is exactly canceled by integrals of momentum space drift and diffusion terms. The latter usually does not occur in thermodynamic equilibrium, as it implies all macroscopic flows stop.

According to these we find: The Maxwell-Boltzmann distribution $f_0(\mathbf{x},\mathbf{p})= C\exp\left(-E_{tot}(\mathbf{x},\mathbf{p})/(kT)\right)$ is an equilibrium solution of the Fokker-Planck equation, but only under very ideal and restrictive conditions. The system must be in strict thermodynamic equilibrium and exchange energy with a constant temperature T heat bath. It must also satisfy the detailed balance condition, meaning that the net particle flux $\mathbf{J}_0$ is zero everywhere in phase space, particularly the momentum space balance condition $\mathcal{F}^i f_0 = K^{ij} \frac{\partial f_0}{\partial p^j}$. Additionally, the total deterministic force $\mathcal{F}^i$ acting on particles must satisfy the generalized Einstein relation $\mathcal{F}^i=-\beta K^{ij} v^j$, where K^{ij} is the momentum space diffusion coefficient and v^j is the particle velocity. This condition requires that the effects of all non-conservative forces (such as magnetic forces) can be exactly canceled, or that the diffusion/friction process has a very precise relationship to temperature to maintain this balance. Furthermore, there must be no external source-sink term $S=0$, and no macroscopic spatial particle flow, meaning that $v^i \frac{\partial f_0}{\partial x^i}=0$, typically by assuming spatial uniformity or a distribution that reaches equilibrium under potential action. Under these ideal conditions, the Fokker-Planck equation indeed describes how the particle system evolves and reaches thermodynamic equilibrium. However, in more general physical scenarios, especially with strong magnetic fields, non-conservative forces, or complex dissipation mechanisms not in strict equilibrium with the heat bath, the steady-state (or quasi-steady-state) distribution of the equation is usually no longer a simple Maxwell-Boltzmann form, possibly requiring numerical methods or analytical solutions under specific simplifications.

8. Conclusion

In this work, we have presented a detailed derivation and analysis of the Fokker-Planck equation for charged particles in a relativistic plasma, set within a general relativistic framework. By extending the classical Liouville equation to include stochastic perturbations, we derived the Fokker-Planck equation, emphasizing its applicability to plasma dynamics and cosmic ray propagation. We systematically explored the mathematical properties of the equation, including its classification as a parabolic partial differential equation, and proved the existence, uniqueness, and stability of weak solutions. Through this analysis, we highlighted the fundamental role of the diffusion and drift terms in describing particle transport under the influence of both deterministic forces, such as gravity and electromagnetism, and random interactions with the background medium. We also derived key conservation laws, providing insights into the behavior of particle number, momentum, and energy in such systems, as well as the conditions under which these quantities are conserved or dissipated. Finally, the equilibrium distribution in the system was shown to converge to a form analogous to the Maxwell-Boltzmann distribution under specific conditions, linking this relativistic plasma model to well-known statistical mechanics frameworks. Our study provides a comprehensive understanding of the Fokker-Planck equation in the context of relativistic plasma dynamics, which is crucial for further research in astrophysics, including cosmic ray propagation and the dynamics of interstellar and galactic plasmas.

References

[1] Parker, E. N. The interstellar dynamical theory of cosmic rays. Phys. Rev. 110, 1445- 1459 (1958).
[2] Jokipii, J. R. Cosmic-ray propagation. I. Charged particles in a random magnetic field. Astrophys. J. 146, 480 (1966).
[3] Chandrasekhar, S. Stochastic problems in physics and astronomy. Rev. Mod. Phys. 15, 1-89 (1943).
[4] Webb, G. M. Relativistic transport theory for cosmic rays. Astrophys. J. 296, 319-330 (1985).
[5] Thorne, K. S. & Blandford, R. D. Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics (Princeton Univ. Press, 2017).
[6] Potekhin, A. Y. Atmospheres and radiating surfaces of neutron stars. Phys. Usp. 57, 735-770 (2014).
[7] Shalchi, A. Nonlinear cosmic ray diffusion theories. Astrophys. Space Sci. Libr. 362 (2009).

[8] Yan, H. Cosmic ray transport in turbulent fields. Living Rev. Comput. Astrophys. 1, 3 (2015).
[9] Ptuskin, V. S. Propagation model for cosmic rays in the galaxy. Astrophys. Space Sci. 272, 95-104 (2000).
[10] Strong, A. W., Moskalenko, I. V. & Ptuskin, V. S. Cosmic-ray propagation and interactions in the galaxy. Annu. Rev. Nucl. Part. Sci. 57, 285-327 (2007).
[11] Dermer, C. D. & Menon, G. High Energy Radiation from Black Holes: Gamma Rays, Cosmic Rays, and Neutrinos (Princeton Univ. Press, 2009).
[12] Longair, M. S. High Energy Astrophysics (Cambridge Univ. Press, 2011).
[13] Waxman, E. Gamma-ray bursts and cosmic ray origin. Astrophys. J. Lett. 452, L1-L4 (1995).
[14] Vietri, M. Foundations of high-energy astrophysics. Astrophys. J. 661, L13-L16 (2007).
[15] Risken, H. The Fokker-Planck Equation: Methods of Solution and Applications (Springer, 1989).
[16] Pavliotis, G. A. Stochastic Processes and Applications: Diffusion Processes, the FokkerPlanck and Langevin Equations (Springer, 2014).
[17] Arnold, V. I., Kozlov, V. V. & Neishtadt, A. I. Mathematical Aspects of Classical and Celestial Mechanics (Springer, 2006).
[18] Lions, P.-L. Mathematical Topics in Fluid Mechanics: Volume 2: Compressible Models (Oxford Univ. Press, 1998).
[19] Cercignani, C. The Boltzmann Equation and Its Applications (Springer, 1988).
[20] Villani, C. A review of mathematical topics in collisional kinetic theory. Handb. Math. Fluid Dyn. 1, 71-74 (2002).
[21] Lions, P.-L. & Toscani, G. Diffusive limits for finite velocity Boltzmann kinetic models. Rev. Mat. Iberoam. 13, 473-513 (1997).
[22] Desvillettes, L. & Villani, C. On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: The linear Fokker-Planck equation. Commun. Pure Appl. Math. 54, 1-42 (2001).
[23] Boltzmann, L. Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen. Sitzungsber. Akad. Wiss. 66, 275-370 (1872).
[24] Maxwell, J. C. On the dynamical theory of gases. Philos. Trans. R. Soc. Lond. 157, 49-88 (1867).
[25] Synge, J. L. The Relativistic Gas (North-Holland, 1957).
[26] Israel, W. Relativistic kinetic theory of a simple gas. J. Math. Phys. 4, 1163-1181 (1963).
[27] Dermer, C. D. & Böttcher, M. Synchrotron radiation from shock waves in active galactic nuclei. Astrophys. J. 295, 599-610 (1985).
[28] Meszaros, P. Theories of gamma-ray bursts. Annu. Rev. Astron. Astrophys. 40, 137-169 (2002).
[29] Blandford, R. D. & Eichler, D. Particle acceleration at astrophysical shocks: A theory of cosmic ray origin. Phys. Rep. 154, 1-75 (1987).
[30] Bell, A. R. Turbulent amplification of magnetic field and diffusive shock acceleration of cosmic rays. Mon. Not. R. Astron. Soc. 353, 550-558 (2004).
[31] Liboff, R. L. *Kinetic Theory: Classical, Quantum, and Relativistic Descriptions* (Springer, 2003).
[32] Krall, N. A. & Trivelpiece, A. W. *Principles of Plasma Physics* (McGraw-Hill, 1973).
[33] Weinberg, S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, 1972).
[34] Evans, L. C. *Partial Differential Equations* 2nd edn (American Mathematical Society, 2010).
[35] Glassey, R. T. *The Cauchy Problem in Kinetic Theory* (SIAM, 1996).